The Upper Total Domination Number of a Graph

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Abstract

The total dominating set S in a connected graph G is called a minimal total dominating set if no proper subset of S is a total dominating set of G. The upper total domination number $\gamma_t^+(G)$ of G is the maximum cardinality of a minimal total dominating sets of G. In this paper we discuss some results on upper total domination number. It is shown that for any integer $a \geq 4$, there exists a connected graph G such that $\gamma_t(G) = a$ and $\gamma_t^+(G) = 2a - 4$.

Keywords : total domination number, minimal total dominating set, upper total domination number.

Mathematics Subject Classification : 05C69

Field : Graph Theory ; Subfield : Domination

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges . For basic definitions and terminologies we refer to [1]. A *total dominating set* (TDS) of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S . Total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi [2] and is well studied in graph theory (see, for example [3, 4]). Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TDS. Total domination is undefined for graphs with isolated vertices. The total dominating set S in a connected graph G is called a *minimal total dominating set* if no proper subset of S is a total dominating set of G. The *upper total domination number* $\gamma_t^+(G)$ of G is the maximum cardinality of a minimal total dominating sets of G. Terms not defined in the paper are used in the sense of Harary [1]. For domination parameters we refer to [5].

2. The upper total domination number

Example 2.1 For the graph G given in Figure 1, $S_1 = \{v_2, v_4, v_5\}$ and $S_2 = \{v_3, v_4, v_5\}$ are the minimum total dominating sets of G so that $\gamma_t(G) = 3$. The set $S = \{v_1, v_3, v_5, v_6\}$ is a total dominating set of G and it is clear that no proper subset of S is the total dominating set of G and so S is a minimal total dominating set of G. Also it is easily verified that no five element or six element subset is a minimal total dominating set of G, it follows that $\gamma_t^+(G) = 4$.

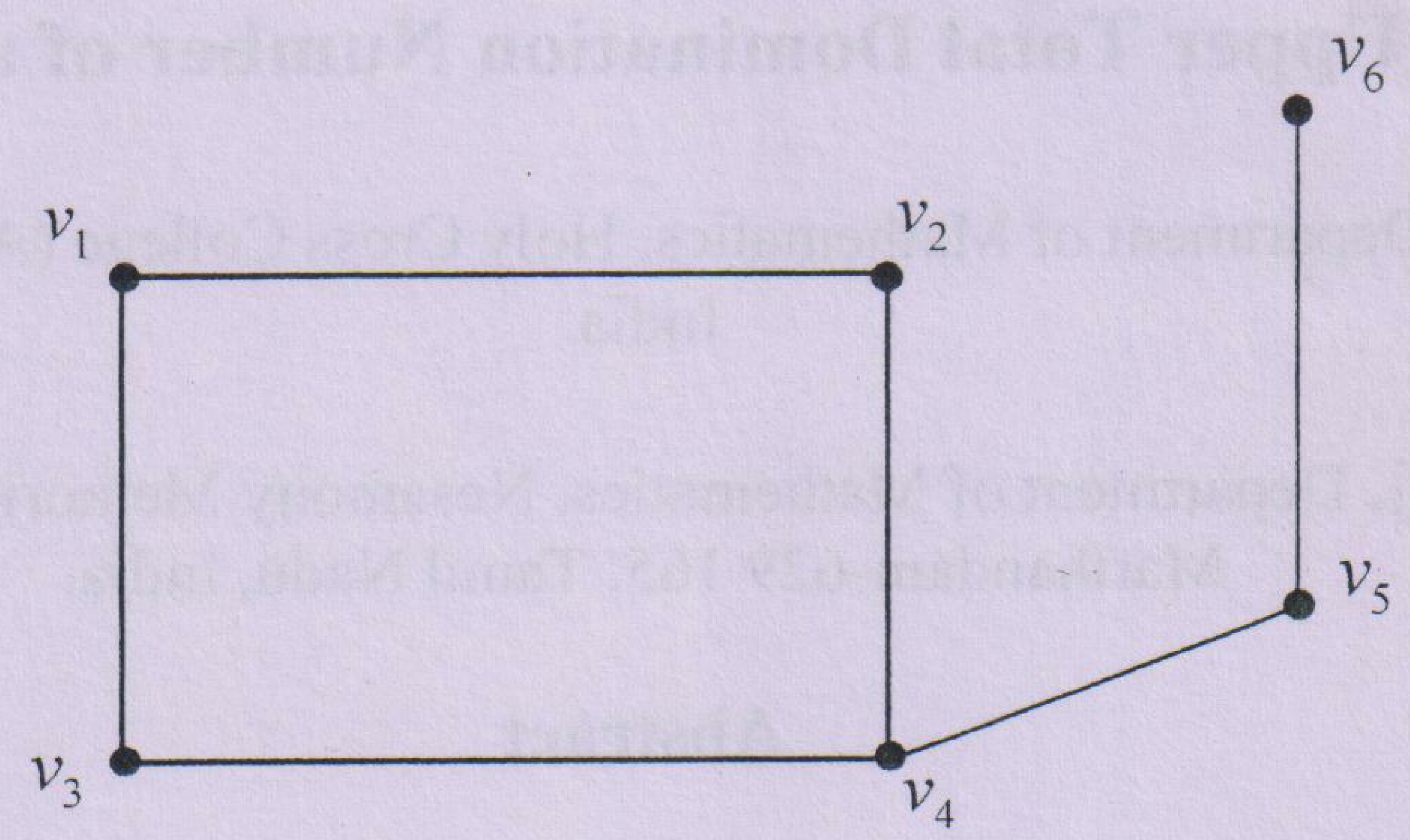

Figure 1

Remark 2.2 Every minimum total dominating set of G is a minimal total dominating set of G and the converse is not true. For the graph G given in Figure 1, $S=\{v_1,v_3,v_5,v_6\}$ is a minimal total dominating set but not a minimum total dominating set of G.

Theorem 2.3 For a connected graph G, $2\le\gamma_t(G)\le\gamma_t^+(G)\le n$.

Proof: We know that any total dominating set needs at least two vertices and so $\gamma_t(G)\ge 2$. Since every minimal total dominating set is also a total dominating set, $\gamma_t(G)\le\gamma_t^+(G)$. Also, since $V(G)$ is a total dominating set of G, it is clear that $\gamma_t^+(G)\le n$. Thus $2\le\gamma_t(G)\le\gamma_t^+(G)\le n$.

Remark 2.4 The bounds in Theorem 2.3 are sharp. For any graph $G=K_n$, $\gamma_t(G)=2$ and $\gamma_t^+(G)=2$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 1, $\gamma_t(G)=3$, $\gamma_t^+(G)=4$ and $n=6$ so that $2<\gamma_t(G)<\gamma_t^+(G)<n$.

Theorem 2.5 For a connected graph G, $\gamma_t(G)=n$ if and only if $\gamma_t^+(G)=n$.

Proof: Let $\gamma_t^+(G)=n$. Then $S=V(G)$ is the unique minimal total dominating set of G. Since no proper subset of S is a total dominating set, it is clear that S is the unique γ_t- set of G and so $\gamma_t(G)=n$. The converse follows from Theorem 2.3.

Theorem 2.6 For any integer $a\ge 4$, there exists a connected graph G such that $\gamma_t(G)=a$ and $\gamma_t^+(G)=2a-4$.

Proof: Let $P_i : x_i, y_i\ (1\le i\le a-3)$ be a path of order 2. Let $C_5 : v_1,v_2,v_3,v_4,v_5,v_1$. Let G be a graph obtained from $P_i\ (1\le i\le a-3)$ and C_5 by joining v_1 with each $x_i\ (1\le i\le a-3)$. The graph G is shown in Figure 2.

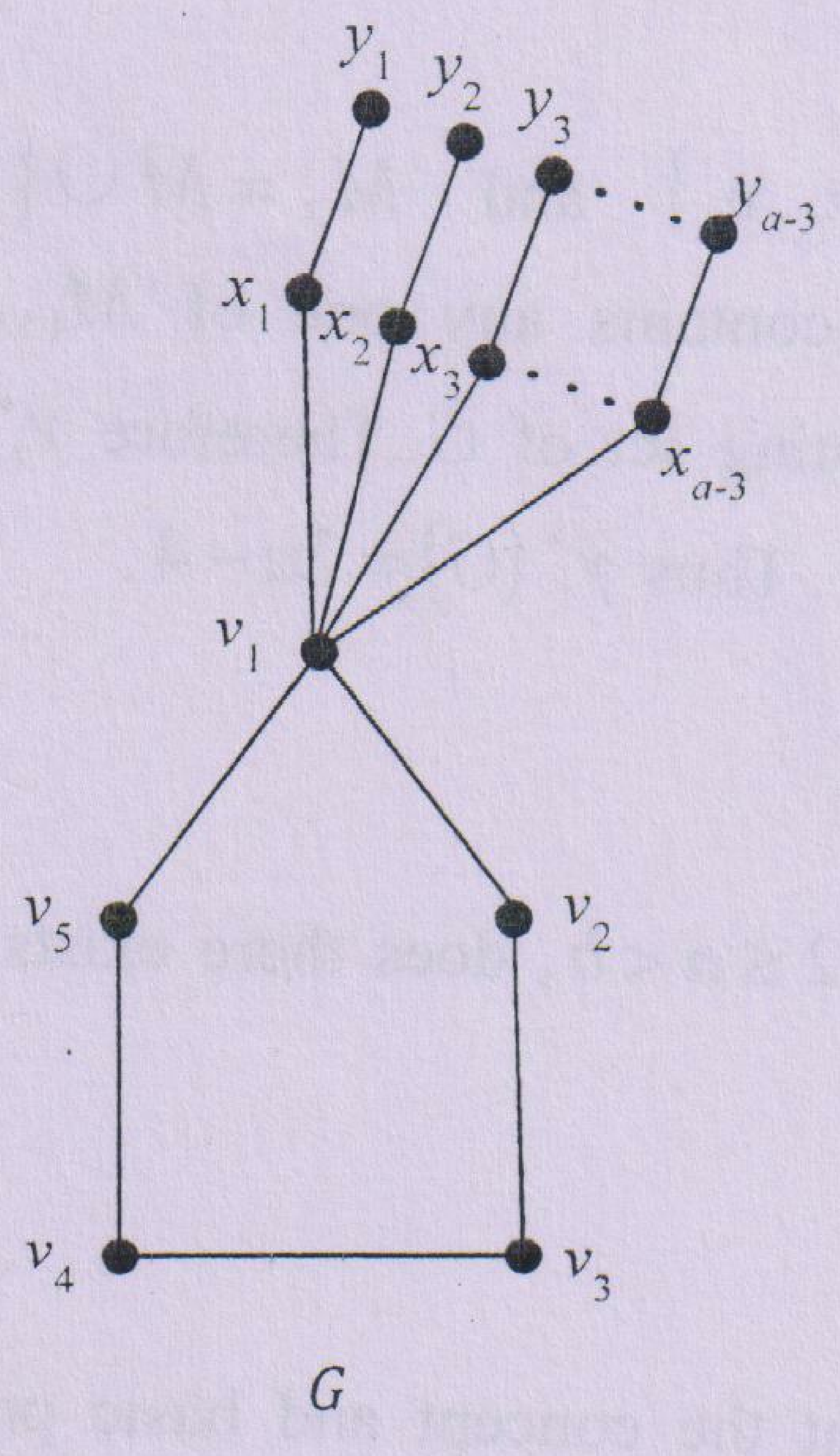

Figure 2

First we claim that $\gamma_t(G)=a$. Let $X=\{v_1,x_1,x_2,...,x_{a-3}\}$. It is easily observed that X is a subset of every minimum total dominating set of G and so $\gamma_t(G)\geq a-3+1=a-2$. It is easily verified that $X\cup\{x\}; x\notin X$ is not a total dominating set of G and so $\gamma_t(G)\geq a$. Now $S_1=X\cup\{v_2,v_3\}$, $S_2=X\cup\{v_3,v_4\}$ and $S_3=X\cup\{v_4,v_5\}$ are the minimum total dominating sets of G so that $\gamma_t(G)=a$.

Next we show that $\gamma_t^+(G)=2a-4$. Now $D=\{x_1,x_2,...,x_{a-3},y_1,y_2,...,y_{a-3},v_3,v_4\}$ is a total dominating set of G. We show that D is a minimal total dominating set of G. Let D' be any proper subset of D. Then there exists at least one vertex say $v\in D$ such that $v\notin D'$. Suppose that $v=x_i$ for some i $(1\leq i\leq a-3)$. Then the vertex y_i $(1\leq i\leq a-3)$ will be isolate in $\langle D'\rangle$. Therefore D' is not a total dominating set of G. Now, assume that $v=y_i$ for some i $(1\leq i\leq a-3)$. Then the vertex x_i $(1\leq i\leq a-3)$ will be isolate in $\langle D'\rangle$ and so D' is not a total dominating set of G. Now, assume that $v=v_3$ or v_4. Then the vertex v_4 or v_3 will be isolate in $\langle D'\rangle$ and so D' is not a total dominating set of G. Therefore any proper subset of D is not a total dominating set of G. Hence D is a minimal total dominating set of G and so $\gamma_t^+(G)\geq 2a-4$. We show that $\gamma_t^+(G)=2a-4$. Suppose that there exists a minimal total dominating set T of G such that $|T|\geq 2a-3$. Then $|T|$ is either $2a-3$ or $2a-2$. Let $|T|=2a-3$. Suppose that $v_1\notin T$. Since $\langle T\rangle$ has no isolated vertex, $x_i,y_i\in T$ for every i $(1\leq i\leq a-3)$. Let $S'=\{x_1, x_2, ..., x_{a-3}, y_1, y_2, ..., y_{a-3}\}$. Since $S_1=S'\cup\{v_2, v_3\}$ $S_2=S'\cup\{v_3,v_4\}$ and $S_3=S'\cup\{v_4,v_5\}$ are total dominating sets of G and since $S'\subseteq T$, it follows that T contains either S_1, S_2 or S_3 and so T is not a minimal total dominating set of G, which is a contradiction. Suppose that $v_1\in T$. Then T consists of $M=\{x_1,x_2,...,x_{a-3}\}$. Since

$M_1 = M \cup \{v_1, v_2, v_3\}$, $M_2 = M \cup \{v_1, v_5, v_4\}$ and $M_3 = M \cup \{v_1, v_3, v_4\}$ are total dominating sets of G, it follows that T contains any one of M_1, M_2, M_3, which is a contradiction to T is a minimal total dominating set of G. Therefore $\gamma_t^+(G) \neq 2a-3$. By the similar argument we can prove $\gamma_t^+(G) \neq 2a-2$. Thus $\gamma_t^+(G) = 2a-4$.

Open Problem 2.7

For every pair a, b of integers with $2 \leq a < b$, does there exists a connected graph G such that $\gamma_t(G) = a$ and $\gamma_t^+(G) = b$?

3. Conclusion

This study is undertaken to highlight the concept and basic properties of upper total domination number.

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